

COBORDISMS OF FOLD MAPS OF 4-MANIFOLDS INTO THE SPACE

BOLDIZSÁR KALMÁR

ABSTRACT. We compute the oriented cobordism group of fold maps of 4-manifolds into \mathbb{R}^3 with all the possible restrictions (and also with no restriction) to the singular fibers. We also give geometric invariants which describe completely the cobordism group of fold maps.

1. INTRODUCTION

Folds maps on oriented 4-manifolds into \mathbb{R}^3 can be considered as additional structures on 4-manifolds like framings of the stable tangent bundle of the 4-manifold [3], surfaces with special properties embedded into the 4-manifold [32], and conditions about the (co)homologies of the 4-manifold [25, 32]. Like in the case of other additional structures, e.g. spin structures, one can define a corresponding notion of cobordism, i.e. cobordism of fold maps of 4-manifolds into \mathbb{R}^3 . Regarding the applications of cobordisms of 4-manifolds equipped with additional structures, it seems to be useful to consider the cobordisms of fold maps of 4-manifolds into \mathbb{R}^3 with prescribed singular fibers. Computing these cobordism groups is the goal of the present paper.

In [17], we computed the oriented cobordism group of fold maps of 3-manifolds into the plane, and by checking the values of the geometric cobordism invariants defined in [19, 20] on the generators, we obtain that the fold cobordism class of a fold map of a 3-manifold into the plane is described by the cobordism classes of the immersions obtained by restricting the fold map to its definite and indefinite singular sets, respectively [19]. In this paper, we obtain that the fold cobordism class of a 4-manifold into \mathbb{R}^3 is determined by the values of the geometric invariants corresponding to the singular set of the fold map, similarly to the 3-dimensional case. Moreover, we obtain a clear and complete picture about the cobordism groups of fold maps of 4-manifolds with prescribed singular fibers. Some of the results of the present paper were obtained in [15]. We proved [19] that the cobordism classes of simple fold maps of $(n + 1)$ -dimensional manifolds into an n -dimensional manifold are described by the geometric invariants defined in [19, 20]. Sadykov proved [28] that the cobordism classes of fold maps of 5-manifolds into \mathbb{R}^4 are described by the geometric invariants defined in [19, 20].

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1.1. Notations. In this paper the symbol “ \amalg ” denotes the disjoint union, γ^1 denotes the universal line bundle over $\mathbb{R}P^\infty$, ε_X^1 (shortly ε^1) denotes the trivial line bundle over the space X , and the symbols ξ^k , η^k , etc. usually denote k -dimensional real vector bundles. The symbols $\det \xi^k$ and $T\xi^k$ denote the determinant line bundle and the Thom space of

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the bundle ξ^k , respectively. The symbol $\text{Imm}_N^{\xi^k}(n-k, k)$ denotes the cobordism group of k -codimensional immersions into an n -dimensional manifold N whose normal bundles are induced from ξ^k (this group is isomorphic to the group $\{\dot{N}, T\xi^k\}$, where \dot{N} denotes the one point compactification of the manifold N and the symbol $\{X, Y\}$ denotes the group of stable homotopy classes of continuous maps from the space X to the space Y). The symbol $\text{Imm}^{\xi^k}(n-k, k)$ denotes the cobordism group of k -codimensional immersions into \mathbb{R}^n whose normal bundles are induced from ξ^k (this group is isomorphic to $\pi_n^s(T\xi^k)$). The symbol $\text{Imm}_N(n-k, k)$ denotes the cobordism group $\text{Imm}_N^{\gamma^k}(n-k, k)$ where γ^k is the universal bundle for k -dimensional real vector bundles and N is an n -dimensional manifold. The symbol $\pi_n^s(X)$ (π_n^s) denotes the n th stable homotopy group of the space X (resp. spheres). The symbol “ id_A ” denotes the identity map of the space A . The symbol ε denotes a small positive number. All manifolds and maps are smooth of class C^∞ .

2. PRELIMINARIES

2.1. Fold maps. Let Q^{n+1} and N^n be smooth manifolds of dimensions $n+1$ and n respectively. Let $p \in Q^{n+1}$ be a singular point of a smooth map $f: Q^{n+1} \rightarrow N^n$. The smooth map f has a *fold singularity* at the singular point p if we can write f in some local coordinates around p and $f(p)$ in the form

$$f(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n-1}, x_n^2 \pm x_{n+1}^2).$$

A smooth map $f: Q^{n+1} \rightarrow N^n$ is called a *fold map* if f has only fold singularities.

A smooth map $f: Q^{n+1} \rightarrow N^n$ has a *definite fold singularity* at a fold singularity $p \in Q^{n+1}$ if we can write f in some local coordinates around p and $f(p)$ in the form

$$f(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n-1}, x_n^2 + x_{n+1}^2),$$

otherwise f has an *indefinite fold singularity* at the fold singularity $p \in Q^{n+1}$. Let $S_1(f)$ denote the set of indefinite fold singularities of f in Q^{n+1} and $S_0(f)$ denote the set of definite fold singularities of f in Q^{n+1} . Let S_f denote the set $S_0(f) \cup S_1(f)$. Note that the set S_f is an $(n-1)$ -dimensional submanifold of the manifold Q^{n+1} . Usually without mentioning we suppose that the source manifold Q^{n+1} and the target manifold N^n are oriented.

If $f: Q^{n+1} \rightarrow N^n$ is a fold map in general position, then the map f restricted to the singular set S_f is a general positional codimension one immersion into the target manifold N^n .

Since every fold map is in general position after a small perturbation, and we study maps under the equivalence relations *cobordism* (see Definitions 2.2), in this paper we can restrict ourselves to studying fold maps which are in general position. Without mentioning we suppose that a fold map f is in general position.

Definition 2.1. For an integer $k > 0$ a fold map $f: Q^{n+1} \rightarrow N^n$ is called a *k-simple fold map* if every connected component of an arbitrary fiber of the map f contains at most k singular points. (For $k = 1$ we say shortly *simple fold map*.)

2.2. Equivalence relations of fold maps.

Definition 2.2. (Cobordism) Two fold maps $f_i: Q_i^{n+1} \rightarrow N^n$ ($i = 0, 1$) of closed (oriented) $(n+1)$ -dimensional manifolds Q_i^{n+1} ($i = 0, 1$) into an n -dimensional manifold N^n are (*oriented*) *cobordant* if

- a) there exists a fold map $F: X^{n+2} \rightarrow N^n \times [0, 1]$ of a compact (oriented) $(n+2)$ -dimensional manifold X^{n+2} ,
- b) $\partial X^{n+2} = Q_0^{n+1} \amalg (-)Q_1^{n+1}$ and
- c) $F|_{Q_0^{n+1} \times [0, \varepsilon]} = f_0 \times \text{id}_{[0, \varepsilon]}$ and $F|_{Q_1^{n+1} \times (1-\varepsilon, 1]} = f_1 \times \text{id}_{(1-\varepsilon, 1]}$, where $Q_0^{n+1} \times [0, \varepsilon]$ and $Q_1^{n+1} \times (1-\varepsilon, 1]$ are small collar neighbourhoods of ∂X^{n+2} with the identifications $Q_0^{n+1} = Q_0^{n+1} \times \{0\}$ and $Q_1^{n+1} = Q_1^{n+1} \times \{1\}$.

We call the map F a *cobordism* between f_0 and f_1 .

This clearly defines an equivalence relation on the set of fold maps of closed (oriented) $(n+1)$ -dimensional manifolds into an n -dimensional manifold N^n .

We denote the set of fold (oriented) cobordism classes of fold maps of closed (oriented) $(n+1)$ -dimensional manifolds into an n -dimensional manifold N^n (into the Euclidean space \mathbb{R}^n) by $\text{Cob}_f^{(O)}(N^n)$ (by $\text{Cob}_f^{(O)}(n)$). We note that we can define a commutative semigroup operation in the usual way on the set of cobordism classes $\text{Cob}_f^{(O)}(N^n)$ by the disjoint union. In the case of $N^n = \mathbb{R}^n$ this semigroup operation is equal to the usual group operation, i.e., the far away disjoint union.

We can refine this equivalence relation by considering the singular fibers (see, for example, [23, 33, 34, 44]) of a fold map.

Definition 2.3. Let τ be a set of singular fibers. Two fold maps $f_i: Q_i^{n+1} \rightarrow N^n$ ($i = 0, 1$) with singular fibers in the set τ of closed (oriented) $(n+1)$ -dimensional manifolds Q_i^{n+1} ($i = 0, 1$) into an n -dimensional manifold N^n are (*oriented*) τ -*cobordant* if they are (oriented) cobordant in the sense of Definition 2.2 by a fold map $F: X^{n+2} \rightarrow N^n \times [0, 1]$ whose singular fibers are in the set τ .

We denote the set of τ -cobordism classes of fold maps with singular fibers in the set τ by $\text{Cob}_\tau^{(O)}(N^n)$.

In this way for an integer $k > 0$ we can obtain the notion of *k-simple fold cobordism* of *k-simple fold maps*, i.e., let τ_k be the set of all the singular fibers which have at most k singular points in each of the connected components of their fibers. We denote the set of *k-simple fold* (oriented) cobordism classes of *k-simple fold maps* of closed (oriented) $(n+1)$ -dimensional manifolds Q^{n+1} into an n -dimensional manifold N^n by $\text{Cob}_{s(k)}^{(O)}(N^n)$. For results about simple fold maps, see, for example, [19, 29, 30, 35, 47].

2.3. Cobordism invariants of fold maps. In [19, 20] for a fold map $f: Q^{n+1} \rightarrow N^n$ we defined homomorphisms

$$\xi_{\text{indef}, n}^N: \text{Cob}_f^O(N^n) \rightarrow \text{Imm}_N^{\det(\gamma^1 \times \gamma^1)}(n-1, 1)$$

and

$$\xi_{\text{def}, n}^N: \text{Cob}_f^O(N^n) \rightarrow \text{Imm}_N^{\varepsilon^1 \times \mathbb{R}P^\infty}(n-1, 1)$$

by mapping a cobordism class of a fold map f into the cobordism classes of the immersions of its indefinite and definite singular set $S_1(f)$ and $S_0(f)$, respectively, with normal bundles induced from the bundle $\eta_{\text{indef}}^1: \det(\gamma^1 \times \gamma^1) \rightarrow \mathbb{R}P^\infty \times \mathbb{R}P^\infty$ and the bundle $\eta_{\text{def}}^1: \varepsilon^1 \rightarrow \mathbb{C}P^\infty$, respectively, where the bundle $\eta_{\text{indef}}^1: \det(\gamma^1 \times \gamma^1) \rightarrow \mathbb{R}P^\infty \times \mathbb{R}P^\infty$ and the bundle $\eta_{\text{def}}^1: \varepsilon^1 \rightarrow \mathbb{C}P^\infty$ are the targets of the universal indefinite and definite germ bundles (see [19, 20]). By [19] we have a homomorphism

$$\theta_n^N \circ \xi_{\text{indef}, n}^N: \text{Cob}_f^O(N^n) \rightarrow \text{Imm}_N(n-1, 1) \oplus \text{Imm}_N^{\gamma^1 \times \gamma^1}(n-2, 2)$$

which maps a fold cobordism class $[f]$ into the sum of the cobordism class of the immersion of the indefinite singular set $S_1(f)$ and the cobordism class of the “twisting” of the indefinite germ bundle over it.

In this paper let us denote shortly the homomorphisms $\theta_n^N \circ \xi_{\text{indef},n}^N$ and $\xi_{\text{def},n}^N$ by ι and δ , respectively. Summarizing, we have a homomorphism

$$\delta \oplus \iota: \text{Cob}_f^O(N^n) \rightarrow \text{Imm}_N^{\varepsilon^1}(\text{cP}^\infty)(n-1, 1) \oplus \text{Imm}_N(n-1, 1) \oplus \text{Imm}_N^{\gamma^1 \times \gamma^1}(n-2, 2)$$

which can be interpreted as a homomorphism $\delta^{\text{imm}} \oplus \delta^{\text{tw}} \oplus \iota^{\text{imm}} \oplus \iota^{\text{tw}}$ into

$$\text{Imm}_N^{\varepsilon^1}(n-1, 1) \oplus \{\dot{N}, \text{SCP}^\infty\} \oplus \text{Imm}_N(n-1, 1) \oplus \text{Imm}_N^{\gamma^1 \times \gamma^1}(n-2, 2)$$

where the homomorphisms δ^{imm} and ι^{imm} map a fold cobordism class $[f]$ into the cobordism class of the immersion of the definite and indefinite singular set of f , respectively, and the homomorphisms δ^{tw} and ι^{tw} map a fold cobordism class $[f]$ into the cobordism class of the “twisting” of the germ bundle over the immersion of the definite and indefinite singular set, respectively.

2.4. Fiber-singularities, Bundle structure and punctured fold maps. Throughout the paper, we use the notions and statements of [17, 16, 19] about (punctured) (multi)fiber-singularities [19], bundle structures of fold maps [19], puncturing fold maps [17], and Pontryagin-Thom type construction for fold maps [16] and for Stein factorizations [17]. By these tools it is enough to deal with cobordisms of punctured Stein factorizations with the appropriate symmetry groups (symmetries of Stein factorizations of fiber-singularities, which come from symmetries of fiber-singularities).

2.5. Symmetry groups of the fiber-singularities. Recall [19] that the symmetry group $\text{ISO}^O(\sigma_{\mathfrak{F}})$ of a punctured indefinite fiber-singularity $\sigma_{\mathfrak{F}}$ can be reduced to a finite group which can be determined by the symmetries of the fiber $\sigma_{\mathfrak{F}}^{-1}(0)$.

Proposition 2.4. *The symmetry groups of the punctured indefinite fiber-singularities are the following. $\text{ISO}^O(\sigma_{\text{I}^1}) = \mathbb{Z}_2$, $\text{ISO}^O(\sigma_{\text{II}^2}) = \mathbb{Z}_2$, $\text{ISO}^O(\sigma_{\text{II}^3}) = D_4$, $\text{ISO}^O(\sigma_{\text{III}^4}) = \mathbb{Z}_2$, $\text{ISO}^O(\sigma_{\text{III}^5}) = \mathbb{Z}_3$, $\text{ISO}^O(\sigma_{\text{III}^6}) = D_3$, $\text{ISO}^O(\sigma_{\text{III}^7}) = \{0\}$, $\text{ISO}^O(\sigma_{\text{III}^8}) = \mathbb{Z}_3 \oplus \mathbb{Z}_2^3$.*

3. MAIN RESULTS

By Section 2.3, we have a homomorphism

$$\delta \oplus \iota = \delta^{\text{imm}} \oplus \delta^{\text{tw}} \oplus \iota^{\text{imm}} \oplus \iota^{\text{tw}}: \text{Cob}_f^O(4, -1) \rightarrow \text{Imm}^{\varepsilon^1}(2, 1) \oplus \pi_2^s(\text{CP}^\infty) \oplus \text{Imm}(2, 1) \oplus \text{Imm}^{\gamma^1 \times \gamma^1}(1, 2),$$

i.e., a homomorphism

$$\delta^{\text{imm}} \oplus \delta^{\text{tw}} \oplus \iota^{\text{imm}} \oplus \iota^{\text{tw}}: \text{Cob}_f^O(4, -1) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4$$

(for the group $\text{Imm}^{\gamma^1 \times \gamma^1}(1, 2)$, see Lemma 4.1 below).

Let $c: \text{Imm}^{\varepsilon^1}(2, 1) \rightarrow \text{Imm}(2, 1)$ ($c: \mathbb{Z}_2 \rightarrow \mathbb{Z}_8$) denote the natural inclusion homomorphism.

Theorem 3.1. *The homomorphism $\delta^{\text{imm}} \oplus \delta^{\text{tw}} \oplus \iota^{\text{imm}} \oplus \iota^{\text{tw}}$ gives a complete invariant of the cobordism group $\text{Cob}_f^O(4, -1)$, which is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}$. The isomorphism can be given by the homomorphism*

$$\delta^{\text{imm}} \oplus \frac{c \circ \delta^{\text{imm}} + \iota^{\text{imm}} - (\delta^{\text{tw}}/3 \bmod 8)}{2} \oplus (\delta^{\text{tw}}/3),$$

which coincides with the homomorphism defined by

$$[f: M^4 \rightarrow \mathbb{R}^3] \mapsto \delta^{imm}([f]) \oplus \frac{[f|_{S_f}] - (\sigma(M^4) \bmod 8)}{2} \oplus \sigma(M^4),$$

where $\sigma(M^4)$ denotes the signature of the oriented source manifold M^4 and $[f|_{S_f}] \in \mathbb{Z}_8$ is the cobordism class of the immersion of the singular set of the fold map f into \mathbb{R}^3 .

Remark 3.2. In [17, 19], we showed that the homomorphism $\delta^{imm} \oplus \iota^{imm}$ gives an isomorphism between the fold cobordism group $Cob_f^O(3, -1)$ and \mathbb{Z}_2^2 , where instead of $\delta^{imm} \oplus \iota^{imm}$, we can also choose the homomorphism $\iota^{imm} \oplus \iota^{tw}$ as an isomorphism, as one can see easily by checking the values on the generators.

The statement of Theorem 3.1 includes implicitly the following.

Proposition 3.3. *For a fold map $f: M^4 \rightarrow \mathbb{R}^3$ of a closed oriented 4-manifold M^4 , we have*

$$[f|_{S_f}] \equiv \sigma(M^4) \equiv \delta^{tw}([f]) \pmod{2},$$

$$\text{and } \delta^{tw}([f]) \equiv 0 \pmod{3}.$$

We remark that the first congruence of the above proposition can be deduced by using results of [5, 9] or [45] very easily and the congruence $\sigma(M^4) \equiv \delta^{tw}([f]) \pmod{2}$ follows from [29, 42] as well. The congruence $\delta^{tw}([f]) \equiv 0 \pmod{3}$ is related to [25, 32].

4. COMPUTING COBORDISM GROUPS OF FOLD MAPS ON 4-MANIFOLDS

Lemma 4.1. *The cobordism group $\text{Imm}^{\gamma^1 \times \gamma^1}(1, 2)$ is isomorphic to \mathbb{Z}_4 and the cobordism group $\text{Imm}^{\eta_{\Pi^3}}(1, 2)$ is isomorphic to \mathbb{Z}_4 .*

Proof. A representative of the generator of the group $\text{Imm}^{\gamma^1 \times \gamma^1}(1, 2)$ is a trivially embedded circle into \mathbb{R}^3 whose normal bundle is twisted by 180 degrees as we go once around the circle.

The group $\text{Imm}^{\eta_{\Pi^3}}(1, 2)$ has an epimorphism onto \mathbb{Z}_4 by forgetting the s_{Π^3} bundle of the representatives, and this epimorphism is also injective. A generator of the group $\text{Imm}^{\eta_{\Pi^3}}(1, 2)$ can be represented by an embedded circle with a 180 degree twist in its normal bundle. Details are left to the reader. \square

Now let us prove Theorem 3.1.

Proof of Theorem 3.1. By [15] the 2-simple fold cobordism group $Cob_{s(2)}^O(4, -1)$ is isomorphic to

$$Cob_s^O(4, -1) \oplus \text{Imm}^{\varepsilon^1 \times \gamma^1}(1, 2) \oplus \text{Imm}^{\eta_{\Pi^3}}(1, 2),$$

which is isomorphic to $\mathbb{Z}_2^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

Let us give representatives of generators of this group and compute the values of the homomorphism

$$\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}: Cob_f^O(4, -1) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4$$

on these representatives.

4.1. Generators of the simple fold cobordism group $Cob_s^O(4, -1)$. The group

$$Cob_s^O(4, -1) = \text{Imm}^{\varepsilon^1}(2, 1) \oplus \text{Imm}^{\varepsilon^1 \times \gamma^1}(1, 2)$$

(see [19]) is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the representatives of the two generators $(1, 0)$ and $(0, 1)$ are given by the punctured simple fold maps $f_i: M_i^4 \rightarrow \mathbb{R}^3$ ($i = 1, 2$) which are constructed as follows.

Let $g: T^2 \rightarrow \mathbb{R}^3$ be the immersion of the torus with one unknotted circle as the set of double points in \mathbb{R}^3 , which represents the non-trivial element in $\text{Imm}^{\varepsilon^1}(2, 1)$: i.e., the image $g(T^2)$ is contained in a small tubular neighbourhood of the circle of double points and the intersection of $g(T^2)$ with a normal 2-disk fiber is the standard “figure eight”, which is rotated by 360 degrees as it goes once around the circle of double points. Let M_1^4 be the total space of a trivial s_{11} bundle over the torus T^2 and let f_1 be the fold map which maps the subbundle corresponding to the double points of the “figure eights” in the fibers s_{11} into \mathbb{R}^3 as the immersion g , and maps a fiber s_{11} of this bundle into a fiber of the trivial normal bundle of $g(T^2)$ as the fiber-singularity $\sigma_{11}: s_{11} \rightarrow J$. The construction of the fold map $f_2: M_2^4 \rightarrow \mathbb{R}^3$ is similar, but we choose the immersion $g: T^2 \rightarrow \mathbb{R}^3$ to be the standard embedding, and M_2^4 is the total space of a non-trivial s_{11} bundle over the torus T^2 such that the cobordism class $\iota^{tw}([f])$ is equal to the element of order two in the cobordism group $\text{Imm}^{\gamma^1 \times \gamma^1}(1, 2) = \mathbb{Z}_4$. It is easy to check that the values of the homomorphism

$$\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}: Cob_f^O(4, -1) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4$$

on the fold cobordism classes $[f_1]$ and $[f_2]$ are equal to $(1, 0, 4, 0)$ and $(1, 0, 0, 2)$, respectively.

4.2. Generators of the cobordism group $Cob_{1^1 + \Pi^2}^O(4, -1)$. By [15] this group is isomorphic to

$$Cob_s^O(4, -1) \oplus \text{Imm}^{\varepsilon^1 \times \gamma^1}(1, 2) = \mathbb{Z}_2^2 \oplus \mathbb{Z}_2.$$

Three representatives of its generators are the two representatives f_1 and f_2 of the generators of the simple fold cobordism group and a third fold map $f_3: M_3^4 \rightarrow \mathbb{R}^3$ which is constructed as follows. Let $f'_3: M_3^4 \rightarrow \mathbb{R}^3$ be the composition $h \circ p$ of the total space $p: s_{\Pi^2}^\# \times S^1 \rightarrow J^2 \times S^1$ of a trivial $\sigma_{\Pi^2}: s_{\Pi^2}^\# \rightarrow J^2$ bundle over the circle S^1 and the embedding $h: J^2 \times S^1 \rightarrow \mathbb{R}^3$, where $h(J^2 \times S^1)$ is the regular neighbourhood of the standard circle $h(\{0\} \times S^1)$ in \mathbb{R}^3 twisted by 360 degrees. The modification of the punctured Stein factorization of the fiber-singularity σ_{Π^2} (see [17]) gives us a way to extend the manifold $s_{\Pi^2}^\# \times S^1$ fiberwise to a closed manifold M_3^4 and the map f'_3 to a fold map $f_3: M_3^4 \rightarrow \mathbb{R}^3$ which has only σ_{11} and σ_{Π^2} as indefinite fiber-singularities and whose indefinite singular set is a torus immersed into \mathbb{R}^3 in the same way as the immersion g in the construction of the fold map f_1 . It is an easy exercise to show that the value of the homomorphism $\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}: Cob_f^O(4, -1) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4$ on the cobordism class $[f_3]$ is equal to $(1, 0, 4, 0)$.

4.3. Generators of the 2-simple fold cobordism group. The 2-simple fold cobordism group $Cob_{s(2)}^O(4, -1)$ is isomorphic to

$$Cob_{1^1 + \Pi^2}^O(4, -1) \oplus \text{Imm}^{\eta_{\Pi^3}}(1, 2) = \mathbb{Z}_2^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

The additional generator can be constructed as follows. Similarly to the above let $p: s_{\Pi^3}^\# \times_{\mathbb{Z}_2} S^1 \rightarrow J^2 \times_{\mathbb{Z}_2} S^1$ be the total space of a $\sigma_{\Pi^3}: s_{\Pi^3}^\# \rightarrow J^2$ bundle over the

circle S^1 with an automorphism of the fiber-singularity σ_{II^3} which acts as a 180 degree rotation on J^2 . Let $h: J^2 \times_{\mathbb{Z}_2} S^1 \rightarrow \mathbb{R}^3$ be the standard embedding (as the tubular neighbourhood of $\{0\} \times_{\mathbb{Z}_2} S^1$) into \mathbb{R}^3 . Now let $f_4: M_4^4 \rightarrow \mathbb{R}^3$ be the fold map which we obtain by closing the manifold $s_{II^3}^\# \times_{\mathbb{Z}_2} S^1$ and the map $h \circ p$ using the modification of the punctured Stein factorization of the fiber-singularity σ_{II^3} (see [17]) similarly to the construction of the previous fold map f_3 . The homomorphism $\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}$ takes the value $(0, 0, 2, x)$ for some $x \in \mathbb{Z}_4$ on the class $[f_4]$ in $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4$.

Summerizing, we have four generators $[f_1], \dots, [f_4]$ of the 2-simple fold cobordism group $Cob_{s(2)}^O(4, -1) = \mathbb{Z}_2^3 \oplus \mathbb{Z}_4$ on which the homomorphism $\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}$ takes the values

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 2 \\ 0 & 2 & 0 & x \end{pmatrix},$$

where the column vectors correspond to the basis $[f_1], \dots, [f_4]$.

4.4. Computing the cobordism group $Cob_{s(2)+III^4+III^6}^O(4, -1)$. By the above matrix it is easy to see that the kernel of the homomorphism

$$\begin{aligned} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 &= Cob_{s(2)}^O(4, -1) \xrightarrow{\varphi_3} Cob_{s(2)+III^4+III^6}^O(4, -1) \xrightarrow{\psi} \\ &Cob_f^O(4, -1) \xrightarrow{\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}} \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4, \end{aligned}$$

where the homomorphisms φ_3 and ψ are the natural homomorphisms, coincides with

- (a) the group of order four generated by $(1, 0, 1, 0)$ and $(1, 1, 0, 2)$ if x is a generator of \mathbb{Z}_4 ,
- (b) the group of order two generated by $(1, 0, 1, 0)$ if x is not a generator of \mathbb{Z}_4 .

Now, we show the case (a) holds. The boundaries $\partial\sigma_{III^4}$ and $\partial\sigma_{III^6}$ of the punctured fiber-singularities σ_{III^4} and σ_{III^6} , respectively, are 2-simple fold cobordant to two punctured fold maps $g_1: N_1^3 \rightarrow S^2$ and $g_2: N_2^3 \rightarrow S^2$ whose indefinite singular set is immersed into S^2 as embedded circles and “small figure eights” where the double points of the “small figure eights” can be the image of a singular fiber II^2 and $II^{1,1}$ in the case of the fiber-singularity σ_{III^4} , and II^2 and II^3 in the case of the fiber-singularity σ_{III^6} . Therefore, we have two punctured 2-simple fold maps $g_1: N_1^3 \rightarrow \mathbb{R}^2$ and $g_2: N_2^3 \rightarrow \mathbb{R}^2$ obtained from the boundaries of the punctured fiber-singularities σ_{III^4} and σ_{III^6} respectively, and we have two $(s(2) + III^4 + III^6)$ -null-cobordisms G_1 and G_2 of these fold maps g_1 and g_2 , respectively, obtained from the punctured fiber-singularities σ_{III^4} and σ_{III^6} , respectively. Now let us take the punctured fold maps $g_i \times \text{id}_{S^1}: N_i^3 \times S^1 \rightarrow \mathbb{R}^2 \times S^1$ ($i = 1, 2$), where we consider $\mathbb{R}^2 \times S^1$ as the embedded normal bundle in \mathbb{R}^3 of a standard embedded circle S^1 , whose fiber is twisted by 360 degrees as we go once around the embedded circle S^1 . Let us consider the same construction with the null-cobordisms G_1 and G_2 . In this way, we obtain two $(s(2) + III^4 + III^6)$ -null-cobordant punctured 2-simple fold maps which are not 2-simple null-cobordant and are not 2-simple cobordant to each other (since only one of them contains not-zero $[f_4]$ component). Hence the case (a) holds, i.e., the kernel of $(\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}) \circ \psi \circ \varphi_3$ is generated by the elements $[f_1] + [f_3]$ and $[f_1] + [f_2] + 2[f_4]$. Moreover the kernel of φ_3 is also generated by these elements.

In the following, we show that the homomorphism φ_3 is surjective (this implies together with the above argument that $(\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}) \circ \psi$ is injective and hence

so is ψ). By the Pontryagin-Thom-Szűcs type construction for fold maps [15, 16], we have the exact sequence¹

$$\begin{aligned} \text{Cob}_{s(2)}^O(4, -1) &\xrightarrow{\varphi_3} \text{Cob}_{s(2)+\text{III}^4+\text{III}^6}^O(4, -1) \xrightarrow{\alpha} \pi_3(\Gamma_{s(2)+\text{III}^4+\text{III}^6}, \Gamma_{s(2)}) \xrightarrow{\beta} \\ &\text{Cob}_{s(2)}^O(3, -1) \xrightarrow{\varphi_2} \text{Cob}_{s(2)+\text{III}^4+\text{III}^6}^O(3, -1), \end{aligned}$$

where the last two cobordism groups are isomorphic to \mathbb{Z}_2^4 and \mathbb{Z}_2^2 respectively (see [17]), φ_2 is surjective, and since we can show that the group $\pi_3(\Gamma_{s(2)+\text{III}^4+\text{III}^6}, \Gamma_{s(2)})$ is isomorphic to \mathbb{Z}_2^2 by an argument similar to [17, after Lemma 5.6], we obtain that α is the null-homomorphism, and hence φ_3 is surjective.

Hence the cobordism group $\text{Cob}_{s(2)+\text{III}^4+\text{III}^6}^O(4, -1)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, and a system of representatives of its generators is given by f_1 and f_4 . Note that the homomorphism $(\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}) \circ \psi: \text{Cob}_{s(2)+\text{III}^4+\text{III}^6}^O(4, -1) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4$ is injective.

4.5. Computing the 3-simple fold cobordism group. We obtain that the cobordism group $\text{Cob}_{s(2)+\text{III}^4+\text{III}^5+\text{III}^6+\text{III}^7}^O(4, -1)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \text{Imm}^{\eta_{\text{III}^5}}(0, 3) \oplus \text{Imm}^{\varepsilon^3}(0, 3)$, which is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^2$.

Let \mathcal{C} denote the oriented cobordism group of fold maps of oriented 4-manifolds into \mathbb{R}^3 where two fold maps are cobordant if and only if they are cobordant by a 3-simple fold map $F: W^5 \rightarrow \mathbb{R}^3 \times [0, 1]$ such that the structure group of its σ_{III^8} fiber-singularity bundle can be reduced to the group \mathbb{Z}_3 .

By [15], the cobordism group \mathcal{C} is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^3$. Moreover, we have a natural surjective homomorphism $\psi'': \mathcal{C} \rightarrow \text{Cob}_{s(3)}^O(4, -1)$ and an injective homomorphism $\gamma: \mathcal{C} \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^3$ which goes through the group $\text{Cob}_{s(3)}^O(4, -1)$, i.e., let γ be the homomorphism

$$((\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}) \circ \psi' \circ \psi'') \oplus \pi_{\text{III}^5} \oplus \pi_{\text{III}^7} \oplus \pi_{\text{III}^8},$$

where $\psi': \text{Cob}_{s(3)}^O(4, -1) \rightarrow \text{Cob}_f^O(4, -1)$ is the natural homomorphism and $\pi_{\text{III}^i}: \mathcal{C} \rightarrow \mathbb{Z}$ is the algebraic number of the fiber-singularity σ_{III^i} of a class in \mathcal{C} ($i = 5, 7, 8$). Hence ψ'' is an isomorphism and the 3-simple fold cobordism group $\text{Cob}_{s(3)}^O(4, -1)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^3$.

4.6. Computing the fold cobordism group $\text{Cob}_f^O(4, -1)$. By the Pontryagin-Thom-Szűcs type construction, the fold cobordism group

$$\text{Cob}_f^O(4, -1)$$

is isomorphic to the group $\text{Cob}_{s(3)}^O(4, -1)/G$, where G is the group generated by the boundaries of the punctured fiber-singularities $\sigma_{\mathfrak{F}}$ with $\kappa(\mathfrak{F}) = 4$ classified in [34].

Since $(\delta \oplus \iota) \circ \psi'$ restricted to the direct summand $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is injective, the factorization by the group G can have effect only to the direct summand \mathbb{Z}^3 .

The boundary of $\sigma_{\text{IV}^{10}}$ gives a relation between the generator of the second \mathbb{Z} component of this direct summand \mathbb{Z}^3 (the \mathbb{Z} component corresponding to the fiber-singularity σ_{III^7}) and an element in the direct summand $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. Hence the factorization by the

¹The homotopy exact sequence for the pair $(\Gamma_{s(2)+\text{III}^4+\text{III}^6}, \Gamma_{s(2)})$, where $\Gamma_{s(2)+\text{III}^4+\text{III}^6}$ and $\Gamma_{s(2)}$ are the Pontryagin-Thom-Szűcs type classifying spaces for punctured oriented $(s(2) + \text{III}^4 + \text{III}^6)$ -maps and 2-simple fold maps, respectively.

boundary of $\sigma_{IV^{10}}$ reduces the group $Cob_{s(3)}^O(4, -1) = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^3$ to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}^2$. Similarly, the boundary of $\sigma_{IV^{11}}$ reduces this last group to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}$.

In order to show that the factorization by the boundaries of the other fiber-singularities have no further effect, it is enough to give an injective homomorphism

$$Cob_{s(3)}^O(4, -1)/\{[\partial\sigma_{IV^{10}}], [\partial\sigma_{IV^{11}}]\} \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4,$$

which goes through the fold cobordism group $Cob_f^O(4, -1)$.

By [34] the algebraic number of the fiber-singularity σ_{III^8} is equal to the signature of the source 4-manifold. Since we have already injective invariants on the direct summand $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, we obtain that the fold cobordism group $Cob_f^O(4, -1)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}$.

4.7. Generators of the cobordism group $Cob_f^O(4, -1)$. By the previous proof, a system of generators consists of the cobordism classes $[f_1]$, $[f_4]$, and the cobordism class of the fold map $f: \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2} \rightarrow \mathbb{R}^3$ constructed in [33] with one singular fiber of type III^8 and Boy's surface as the immersion of the indefinite fold singular set. Let \mathfrak{b}_1 , \mathfrak{b}_2 and \mathfrak{b}_3 denote the classes $[f_1]$, $[f_4]$ and $[\tilde{f}]$, respectively, where \tilde{f} denotes the fold map obtained from $f: \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2} \rightarrow \mathbb{R}^3$ by reversing the orientation of the source manifold $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$.

The homomorphism $\delta^{imm} \oplus \delta^{tw} \oplus \iota^{imm} \oplus \iota^{tw}$ takes the values

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 4 & 2 & 1 \\ 0 & x & y \end{pmatrix}$$

on the generators $\mathfrak{b}_1, \mathfrak{b}_2$ and \mathfrak{b}_3 , where the column vectors correspond to the generators $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$, and x is a generator of \mathbb{Z}_4 . Hence, the homomorphism

$$\delta^{imm} \oplus \frac{c \circ \delta^{imm} + \iota^{imm} - (\delta^{tw}/3 \bmod 8)}{2} \oplus (\delta^{tw}/3)$$

is an isomorphism between the groups $Cob_f^O(4, -1)$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}$, where $c: \mathbb{Z}_2 \rightarrow \mathbb{Z}_8$ is the natural inclusion homomorphism.

Moreover, we can choose another homomorphism depending on y as well, which detects the \mathbb{Z}_4 summand of the group $Cob_f^O(4, -1)$, namely the homomorphism $\iota^{tw} - (k(\delta^{tw}/3) \bmod 4)$, where $kx \equiv y \bmod 4$. Hence, for a fold map $g: M^4 \rightarrow \mathbb{R}^3$ of a closed oriented 4-manifold, we have

$$\frac{[f|_{S_f}] - (\sigma(M^4) \bmod 8)}{2} \equiv \iota^{tw}([f]) - k\sigma(M^4) \bmod 4.$$

□

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KYUSHU UNIVERSITY, FACULTY OF MATHEMATICS, 6-10-1 HAKOZAKI, HIGASHI-KU, FUKUOKA 812-8581, JAPAN

E-mail address: kalmbold@yahoo.com